Joint Meeting of UMI-SIMAI / SMAI-SMF
"Mathematics and its Applications"

Panel on Didactics of Mathematics
July, 6th

Dipartimento di Matematica
Università di Torino
Via Carlo Alberto, 10
room IV, 2nd floor

Organisers: Michèle Artigue (Université Paris 7 Denis Diderot)
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Summary
of the invited lectures

Theme 1:
"Technologies and the Teaching of Mathematics"

Colette Laborde

Using technology as an instrument for mediating knowledge in the teaching of mathematics: the case of dynamic geometry

It is usually considered that technology does not change the conceptual aspects of a mathematical activity but just makes easy and fast some technical parts of this activity.
The perspective adopted in this talk is different. It is based on the following assumptions:

- the use of technology introduces changes in the conceptual part of the mathematical activity,
- solving a problem by using technology includes technical aspects different from those required in paper and pencil environment
- technical and conceptual aspects of a mathematical activity are intertwined.

The talk will discuss to what extent teaching and in particular tasks given to students can take advantage of changes introduced by technology in order to organize the learning of new knowledge by students. The discussion will be illustrated with examples based on the use of dynamic geometry coming from various research works.

Maria Alessandra Mariotti

Artefacts and instruments for mediating mathematical meanings

A historical and epistemological analysis shows the intimate relationship relation between the use of particular artefacts (Abacus, Prospectographs, …) and the evolution of mathematical knowledge, i.e. mathematical objects, methods and theories.
Starting from this perspectives new technologies, and in particular the potentialities of their use may be analysed in order to discuss their contribution to mathematics education. The integration of a technical tool in the didactic system is expected to affect the not only the single components of the system, but also the mutual relationship between them, redefining the didactic system globally.
Examples drawn from different research studies will illustrate the complexity of this process. special emphasis will be put on the role of the teacher in organizing didactic activities centred on the use of a technical tool.

Theme 2:
"Ordinary and innovative practices in the classroom of mathematics"

Claire Margolinas
What mathematical knowledge does the teacher need?

This conference is dedicated to the mathematical knowledge that is needed by the teacher as a professional tool. I will try to explore some aspect of this delicate and broad subject.
"Where does the teacher learn mathematics?" is our first question. If we refer to the secondary mathematics teachers, the University seems to be the main place of learning mathematics. Even for secondary mathematics teachers, the transformation of the mathematics they have learn at university into professional tools for teaching is not so obvious. But if we refer to the primary teacher, the question is far more delicate. For the majority of French primary teachers, the basics of mathematics may have been studied in secondary school, since they may not have dealt with any mathematics at university level, in this case, the two years in the IUFM[1] should provide the specific knowledge that the teacher needs at the beginning of his career, but in a very short time.
"When does the teacher need mathematical knowledge?" is our second question. One of these moments happens when the teacher needs to understand the sense of the pupils' answers. I will show some examples taken in the early learning of counting (age 5-7), in order to understand what mathematical knowledge is involved in the understanding of the pupils' answers. The reactions of the teacher can be questioned on the basis of this mathematical knowledge. Another moment of the teacher's activity has to be taken into account: when the teacher conceived and planned the whole subject of counting. Our main question: *"What mathematical knowledge does the teacher need?"* can be documented by this example. The problem that we should address is: *"in which occasion may the teacher learn it?"* Could it be during the initial studies? Could it be during in service training? What should be the sharing of responsibility between the different institutions involved?
We conclude our investigation by turning back to the places of learning, and in particular the secondary school and we consider some consequences on the teaching of mathematics at this level.

Ornella Robutti
Interactions in classroom with technologies: signs and meanings

This lecture is part of a long-term research on the construction of mathematical meanings around infinity, through the interaction with various technologies, with teaching experiments from kindergarten to secondary school.
The concept
The existing research (e.g. Boero *et al*., 2003) underlines the complexity of the conceptualisation of infinity, pointing out its multi-faceted sides. For example, it reveals sensible to textual and contextual aspects, to classroom social interaction situations, to the cultural environment lived by pupils. Historically, many mathematical concepts have been generated speculating on infinite processes and with big jumps between the current ideas in the culture of the time and the new ones. From the epistemological point of view, a persisting conflict exists between two main approaches to infinity: the *potential* and the *actual* one, the first being an ongoing process that never ends and the second a given object (Wallace, 2003).
The teaching experiments

[1] Institut Universitaire de Formation des Maîtres

I discuss three examples of teaching experiments at different school levels: kindergarten, primary school and secondary school. These examples show students' activities in learning contexts: the first is focused on the natural numbers, the second on the fractions and decimal numbers, the third (Arzarello et al., 2004) on the definite integral with symbolic-graphic calculators. The didactical methodology is based on social interaction between students: working groups and class discussions. The use of different artefacts (more or less technological) is part of the so-called *mathematics laboratory* (UMI, 2004), as a set of various and structured activities, aimed at the construction of meanings of mathematical objects, wherever and whenever it is possible, inside or outside school. The mathematics laboratory particularly supports the perceptuo-motor way of learning, which can be integrated with the symbolic-reconstructive way, in order to give new knowledge.
This social interaction among students is possible thanks to different semiotic systems, as artefacts, words, gestures, every kinds of signs that can support and mediate the construction of mathematical meanings.

The analysis

The research methodology is based on analysis of videotaped activities, where students work in small groups or discuss altogether, coordinated by the teacher or the observer, who has a role of participant observer. Analysing these examples, I use an embodied approach integrated by semiotic-cultural elements, according to the recent literature. It seems that more than the Basic Metaphor of Infinity (Lakoff and Núñez, 2000) can be the result of the students' conceptualisation of infinity, starting from various activities in contexts. Some not negligible elements are: the role of the artefacts, of the social interaction, of the teacher and of the culture. Particularly, a research question is about the influence of artefacts on the learning process, at cognitive level, involving not only the schemes of use introduced by the students, but also the mathematical concepts. In fact, the progressive construction of meaning passes through various ingredients: the use of an artefact may influence the introduction of signs in the social interaction, which can be progressively used, changed, refined, enriched with other signs, till reaching a new meaning. So the interest of my research is on the construction of mathematical meanings (Robutti, in press) from a semiotic and cultural perspective (Radford, in press). And according to it, the final aim of the teaching experiments consists in aligning subjective meanings, which can be particular, according to "the very idiosyncratic nature of students' individual conceptions" (Sinclair & Schiralli, 2003), with the cultural ones that must be general and shared by the community of mathematicians.

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Early birds session (summary)

Bettina Pedemonte, Elisabetta Robotti

Competenze acquisite nell'ambito di uno scambio culturale Italia-Francia

La nostra formazione dottorale in Didattica della Matematica si è svolta nell'ambito della convenzione di co-tutela fra Italia e Francia. Tale convenzione ci ha consentito di acquisire una formazione caratterizzata da competenze contenutistiche, metodologiche e tecniche che hanno integrato gli aspetti propri della scuola francese e della scuola italiana.
Attualmente, la nostra ricerca si svolge presso l'Istituto di Tecnologie Didattiche del CNR di Genova, con il quale abbiamo iniziato a collaborare dall' inizio nel 2002.
Com'è naturale pensare, le competenze acquisite durante la formazione dottorale vengono reinvestite nell'attuale lavoro di ricerca anche se qui il focus della ricerca non ha solamente un carattere didattico, ma è fortemente orientato verso l'uso delle tecnologie come strumento di mediazione in ambito educativo. Per questo motivo, le competenze didattiche del nostro bagaglio dottorale vengono integrate e supportate dalle nuove competenze che giorno dopo giorno conquistiamo nel nostro lavoro in ITD.

La collaborazione con l'ITD ha inizio nell'ambito del progetto europeo ITALES (Innovative Teaching and Learning Environments for School, IST-2000- 26356) che si è concluso nel 2004; Obiettivo principale del progetto è stata la creazione di una comunità europea virtuale di insegnamento e apprendimento per la condivisione di contenuti pedagogici fra studenti e insegnanti attraverso diversi strumenti di comunicazione.
Nell'ambito di tale progetto, il gruppo di ricerca dell'ITD è stato impegnato da un lato nella messa a punto della nuova versione del software per il problem solving aritmetico ARI-LAB 2 (http://www.itd.cnr.it/arilab/index.html) e, dall'altro, nella realizzazione e nella sperimentazione di uno scenario d'uso del software ARI-LAB 2.
La realizzazione della nuova versione del software ci ha coinvolte più direttamente nella messa a punto di una nuova interfaccia e nel design di alcune nuove funzioni operative che sono state integrate in diversi Micromondi di ARI-LAB 2. Per la realizzazione dello scenario d'uso, invece, abbiamo investito direttamente le nostre competenze didattiche. Infatti, tale scenario è stato presentato come corso on-line bimodulare: Modulo I "Problem solving aritmetico" e Modulo II "Frazioni". In particolare, le attività del corso sono state progettate per modificare in modo innovativo approcci tradizionali di insegnamento apprendimento del problem solving aritmetico sfruttando le potenzialità rappresentative e operative del sistema ARI-LAB 2.

Dopo il progetto ITALES la nostra collaborazione con l'ITD è proseguita nell'ambito del progetto TELMA (Technology enhanced learning in mathematics), progetto di ricerca supportato dal network di eccellenza Kaleidoscope. TELMA è un progetto della linea "la scuola del futuro, modelli, metodologie e prototipi per l'innovazione educativa". Lo scopo del progetto è quello di raggiungere un alto livello di integrazione fra i team partecipanti che condividono una comune area di interesse. Nel caso specifico, l'area di interesse comune è l'uso della tecnologia nell'educazione matematica.
Il contributo del nostro gruppo di ricerca in TELMA ha avuto inizio con la stesura di un documento che aveva lo scopo di fornire una presentazione integrata dei quadri teorici dei diversi team. In tale documento, denominato IPTA (Integrated Presentation of Teams Approaches), l'integrazione delle

ricerche è stata inizialmente realizzata considerando aspetti ritenuti significativi per la mutua conoscenza ed il confronto: le aree di ricerca e gli obiettivi, i theoretical frameworks, i tools utilizzati o progettati, i contesti e le metodologie di lavoro. Lo scambio di documenti, articoli, etc che riguardavano ciascuno di questi aspetti ha costituito il materiale di base per il lavoro di integrazione. In particolare, per realizzare l'analisi trasversale dei quadri teorici di riferimento dei teams coinvolti nel progetto, si è resa necessaria la costruzione di un framework teorico che avesse un carattere neutrale rispetto ai quadri teorici dei singoli team e che costituisse uno strumento di indagine per definire e confrontare i diversi modi in cui la tecnologia veniva usata come strumento per l'educazione matematica. Il costrutto teorico così prodotto è stato definito "funzionalità didattiche del tool" ed è caratterizzato da tre componenti (l'obiettivo didattico, il tool e le sue modalità d'uso) che consentono di descrivere le modalità d'uso di una particolare tecnologia in ambito didattico per il raggiungimento di un certo obiettivo didattico.
Anche la messa a punto di questo costrutto è stata supportata dalla sinergia della nostra duplice esperienza di ricerca: da un lato, l'esperienza acquisita in ambito ITD, che si concentra in modo particolare nell'analisi di software didattici con un occhio non esclusivamente didattico ma anche progettuale e strutturale, e, dall'altro, l'esperienza di formazione dottorale che ha affinato la capacità di astrarre elementi salienti da quadri teorici diversi.

Questo costrutto teorico ha rivelato aspetti e potenzialità di grande interesse e per questo è stato assunto come strumento teorico di riferimento in un nuovo progetto, ReMath (Representing Mathematics with Digital Media), proposto nell'ambito di TELMA e nel quale sono stati coinvolti gli stessi team di TELMA. Il progetto ha l'obiettivo di integrare gli aspetti teorici di sviluppare DDA (Dinamic Digital Artefacts) per la rappresentazione matematica, sviluppare scenari per il loro uso e produrre cross-experimentation che prevedono la sperimentazione incrociata delle DDA tra i partner.
ReMath è un progetto in progress nel quale il nostro gruppo di ricerca si è impegnato sia nella produzione di una definizione condivisa del concetto di scenario di uso, sia nella progettazione di un nuovo strumento tecnologico, ALNUSET DDA, per l'esplorazione delle proprietà degli insiemi numerici e lo studio dell'algebra mediante un approccio non formale che sfrutta le potenzialità di visualizzazione, interattività e computazione offerte dal computer technology.
In particolare, il nostro contributo nella realizzazione di questo nuovo software ha sia un carattere strutturale (progettazione dell'interfaccia e delle funzioni del software) sia un carattere didattico in quanto le funzioni del software devono incorporare i concetti matematici che l'uso del software dovrebbe poter mediare.

Rispetto alla nostra formazione dottorale il nostro attuale lavoro di ricerca è notevolmente cambiato non solo per quanto riguarda il contenuto della ricerca ma anche l'ambito in cui essa viene effettuata. Fare ricerca all'interno di un progetto europeo è notevolmente diverso da un lavoro di tesi. Ciò risulta evidente da alcuni fattori quali vincoli e scadenze stabiliti in modo formale che non seguono le esigenze di crescita di un gruppo e che costringono a produrre in tempi brevi. D'altra parte essere partner di un progetto europeo significa tener conto della produzione di altri e questo affina necessariamente le capacità di lavoro collaborativi dando un evidente valore aggiunto al lavoro di ricerca. Infine possiamo dire che la nostra formazione dottorale ha costituito una solida base per poter continuare a crescere nella ricerca ampliando l'ambito delle nostre competenze e offrendo un ambiente fertile per la correlazione tra l'osservazione sperimentale e la costruzione di modelli teorici.

Summary
of the contributed papers

Giovannina Albano, Pier Luigi Ferrari (18.30, room 4)

The impact of e-learning on mathematics education: some experiences at university level

The aim of this proposal is to illustrate a study about the use of some e-learning platforms for the teaching and learning of Mathematics at University level. The development and the spreading of Internet has allowed the access to a great amount of information and resources. The impact on education and training was unavoidable. The e-Learning Programme [3] launched by the European Commission is directed to the development of the potential offered by Internet to reach a greater access to Learning and Training and stresses «*the need for innovative pedagogical approaches and for ambitious objectives regarding learning quality and easy access to e-learning resources and services* ».

Our research concerns the *blended* use of the e-learning platforms (that is as support to face-to-face activities) for the teaching/learning of Mathematics at University level. We are interested both in affective factors and meta-cognitive aspects. The research questions are oriented to understand:

- which are the expectations and the beliefs of the students about e-learning;
- how and through which of the opportunities available the use of an e-learning platform can support the change of:
    - the motivation to learn;
    - the attitudes;
    - the relationship with the Mathematics;
    - the relationship with the teacher;
    - the learning strategies;
- how and through which of the tools available the use of an e-learning platform can create new opportunities for studying and learning.

The current research follows the theoretical principles of constructivism and Jonassen's meaningful learning (2003, [4]). Meaningful learning is characterised by being:

- **active**, that is it makes the student responsible of his/her own results;
- **constructive**, integrating new ideas with prior knowledge in order to make sense or make meaning or reconcile a discrepancy, curiosity, or puzzlement;
- **collaborative**, in particular through the reciprocal teaching and the scaffolding/coaching offered by the teacher;
- **conversational**, because it involves the social processes and in particular the dialogical-argumentative ones;
- **intentional**, since it actively engages the students to achieve cognitive goals;
- **contextualised**, as learning tasks are situated in some meaningful real-world;
- **reflective**, since the students articulate what they have learned reflecting on the processes they develop and the decisions they make.

In this framework the use of an e-learning platform is suitable for constructive and interactive methods, and in particular for the discursive approach to the mathematics. Briefly, a range of opportunities are offered:

- peer-to-peer interaction;
- flexible role of the tutors (plain instructors, agent provocateur, proof readers, teachers, …);
- various semiotic systems: verbal texts, symbolic expressions, figures;
- various cognitive functions of verbal texts: from texts/processes (forum) to texts/objects (workshop);
- the role-play, which is a variant of cooperation and has the function to put the students in active roles and to reverse the attitude toward the problems.

Concerning the affective factors, a first investigation has been carried out at the University of Salerno in order to identify the expectations of the students toward a *blended* course: what is the influence of the ICT on the quality of the course, on the learning at stake, on the relationship with Mathematics and the teacher (Albano, 2005, [1]).

A more refined investigation is in progress taking into account the studies of Zan (1996, [5]) which point out how the change of negative attitude influences success in learning.

Concerning the meta-cognitive aspects, the idea is to support the students by on-line activities, time restricted, based on role-plays, which actively engage them and induce them to face learning topics in a more critical and less mnemonic way. In our setting, the course programme has been split into various parts and each part into as many topics as the number of students engaged. For each part a sequence of activities based on role-play has been created. Each student had to deal with three topics: for the first topic, the student acts as a teacher who wants to evaluate someone other's learning, so he/she has to devise some suitable questions; for the second topic, the student has to answer to the questions prepared by a colleague; finally for the third topic, the students again acts as a teacher, checking the correctness of the work made by other two colleagues. At the end of each sequence, the files produces by the students were revised by the teacher-tutor of the course and the revised files were made available to the students.

A first experimentation has been performed with the Electronic Engineering freshman students of the University of Salerno in 2004-2005. The outcomes have been collected at the end of the course by means of interviews, aimed at understanding how the activities carried out have affected the way of studying, which pros have been noticed by the students themselves, which role (among those played) has been considered particularly useful and why (Albano, 2006, [2]).

This initial idea, related to the problem posing, has been further developed in collaboration with P. L. Ferrari and a parallel experiment has been performed between the University of Salerno and the University of Piemonte Orientale. In the experience at Alessandria the teacher works behind the scenes, and the relations with the students engaged in the experiment are managed by two tutors. Further changes consist in the widening of the role-play to the problems, with a prominent attention to the coordination among semiotic systems (e.g., verbal texts-symbolic expressions-graphs). Moreover a control group has been arranged which is asked to solve problems about theoretical and practical aspects assigned by the staff (teacher+tutors); each student autonomously solve them in a given period of time and at its expiration the staff makes available a solution model for self-evaluation. In other words, the control group uses the platform without performing the role-play.

The research will follow two main directions:

- collection of further data and refinement of the analysis of the cooperative activities based on the platform, in particular concerning the use of language and attitudes towards Mathematics;
- construction and experimentation of personalised remedial paths for students with learning problems, exploiting at best the great flexibility of the tool, which allows very open work modalities (availability of teaching material with final self-evaluation tests) as well as more guided ones (personalised learning paths, self-evaluation questionnaires text comprehension tests and so on).

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Giorgio T. Bagni (16.30, room 1)

Didactics and history of numerical series: Grandi, Leibniz and Riccati, 100 years after Ernesto Cesaro's death

1. Historical introduction: Grandi's series

According to Hans-Georg Gadamer, "*to think historically* actually means *to carry out completely the transposition concepts of the past go through* when we try to think on the basis of them. [This] always implies a mediation between [historical] concepts and our thinking" (Gadamer, 2000, 809-811; in this paper the translations are ours). We are going to apply these ideas in the discussion of some educational possibilities related to an historical example, a well-known indeterminate series.

In 1703, Guido Grandi (1671-1742) noticed that from 1−1+1−1+... it is possible to obtain 0 or 1:

$$(1-1)+(1-1)+(1-1)+\ldots = 0+0+0+\ldots = 0 \qquad 1+(-1+1)+(-1+1)+\ldots = 1+0+0+\ldots = 1$$

The sum of this series was considered ½ by Grandi. According to him, a proof can be based upon the following expansion (expressed by using modern notation), nowadays accepted if and only if $|x| < 1$:

$$\frac{1}{1+x} = \sum_{i=0}^{+\infty}(-x)^i = 1 - x + x^2 - x^3 + ...$$

From $x = 1$ (of course this is *not* correct) we should have: 1−1+1−1+... = ½.
However we have to take into account the following important issue: *did the term "convergence" (with its modern meaning) belong to Grandi's vocabulary?* So could we propose a correct historical analysis of Grandi's series on the basis of the notion of convergence? And what about educational implications?

2. *Series in the 18th century: Leibniz and Riccati*

Gottfried Wilhelm Leibniz (1646-1716) studied Grandi's series in some letters (1713-1716) to German philosopher Christian Wolff (1678-1754), where Leibniz introduced the "probabilistic argument" (that influenced, for instance, Johann and Daniel Bernoulli).
Leibniz noticed that if we "stop" the infinite series 1−1+1−1+... (Leibniz, 1716, 187), it is possible to obtain either 0 or 1 with the same "probability". As a matter of fact, "the *series finita* […] can have an even number of terms, and the final one is negative: 1−1, or 1−1+1−1, or 1−1+1−1+1−1 […] or it can have an odd number of terms, and the final one is positive: 1, or 1−1+1, or 1−1+1−1+1" (Leibniz, 1716, 187). Leibnitian original conclusion is the following: "when numbers' nature vanishes, our possibility to consider even numbers or odd numbers vanishes, too. […] So taking into account what is stated by the authors that wrote about evaluations, […] we ought to take the arithmetic mean [of 0 and 1], i.e. the half of their sum; and in this case nature itself respects *justitiae* law" (Leibniz, 1716, 187). Hence the "most probable" value is the arithmetic mean of 0 and 1, that is ½.
Forty years later, Jacopo Riccati (1676-1754) criticised the convergence of Grandi's series to ½; in his *Saggio intorno al sistema dell'universo* (1754), he wrote: "[Grandi's] argument is interesting, but wrong because it causes contradictions. […] The mistake is caused by the use of a series […] from which it is impossible to get any conclusion. In fact, […] it does not happen that the following terms can be neglected in comparison with preceding terms; this property is verified only for convergent series" (Riccati, 1761, I, 87). In fact, Riccati made reference to some fundamental keywords referred to convergence: his vocabulary is clearly different from Grandi's one.

3. *History and mathematics education*

Let us now briefly consider our students' opinions regarding Grandi's series. A test (Bagni, 2005) has been proposed to students of two third-year *Liceo Scientifico* classes, total 45 students (aged 16-17 years), and of two fourth-year *Liceo scientifico* class, 43 students (aged 17-18 years; total: 88 students), in Treviso (Italy). Their mathematical curricula were traditional: in all classes, at the moment of the test, students did not know infinite series. We asked our students to consider the expression "1−1+1−1+..." (studied "in 1703" by "the mathematician Guido Grandi"), taking into account that "addends, infinitely many, are always +1 and −1" and to express their "opinion about it" (time: 10 minutes; no books or calculators allowed). Some students stated that the sum of the considered series is ½ and they made reference to justifications similar to Leibnitian "probabilistic argument". Audio-recorded material allowed us to point out a salient short passage (1 minute and 35 seconds, 9 utterances):

[1] Researcher: "Why did you write that the result is ½?"
[2] Mirko: "Oh, well, I start with 1, so I have 0, then 1, 0 and so on. There are infinitely many +1 and −1."
[3] Researcher: "That's true, but how can you say ½?"
[4] Mirko: "If I add the numbers, I obtain 1, 0, 1, 0 and always 1 and 0. The mean is ½."
[5] Researcher: "And so?"
[6] Mirko: "The numbers that I find are 1, 0, and 1, 0, and 1, 0 and so on: clearly, for every couple of numbers, one of them is 0 and one of them is 1. So these possibilities are equivalent and their mean is ½."
[7] Mirko: [*after 12 seconds*] "Perhaps my answer is strange, or wrong, but I don't see a different correct result: surely both the results 0 and 1 are wrong. If I say that the result is one of that numbers, for instance 1, I forget all the other numbers, an infinite sequence of 0."
[8] Researcher: "So in your opinion both 0 and 1 cannot be considered the correct answer."
[9] Mirko: "Alright, and in this case what is the result? I wrote that ½ is the results of the operation because ½ is the mean, so it is a number that, in a certain sense, contains both 0 and 1."

Mirko stated that "for every couple of numbers, one of them is 0 and one of them is 1" ([4]) and "the mean […] is a number that, in a certain sense, contains both 0 and 1"([9]). So he did not make explicit reference to the probability: he mainly tried to find a result for the considered problem, and this is an educational issue (influenced by the didactical contract); in the 18th century, the probabilistic argument was based upon a slightly different remark, according to which if we "stop" the infinite series 1−1+1− 1+..., it is possible to obtain both 0 and 1 with the same "probability".
Apart from this difference, what is, nowadays, the correct reaction to be assumed by the teacher? To state "Grandi's series converges" is wrong; but our reaction, as we shall see, would require "irony" (in the sense of: Rorty, 2003, 89-90)

4. Mathematics and irony

Of course Grandi's series is indeterminate; nevertheless it "converges", for instance, in the sense of Georg Frobenius (1849-1917); this notion is based upon ideas of Daniel Bernoulli and Joseph Raabe (1801-1859), and has been generalized by Ludwig Otto Hölder (1859-1937) and Ernesto Cesàro (1859-1906). Given the series $a_0+a_1+a_2+\ldots$, let us consider the sequence: $s_0 = a_0$; $s_1 = a_0+a_1$; $s_2 = a_0+a_1+a_2$ etc. For every $n\geq 0$ let σ_n be the arithmetic mean of $s_0, s_1, \ldots, s_n$. We say that a series converges "according Frobenius-Cesàro" if the sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ converges (usually). With reference to Grandi's series, the sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ is: 1, 1/2, 2/3, 1/2, 3/5, 1/2, 4/7, … and it *converges* (usually) to ½. So the statement "Grandi's series does not converge" could be criticised: it requires "irony", in Richard Rorty's words, i.e. a subject's frame of mind to discuss his/her own vocabulary, and the awareness that this vocabulary is not "closer to reality than others" (Rorty, 2003, 89-90).
Let us remember that in the 20th century many mathematical techniques based upon the "convergence" (summability) according to Frobenius-Cesàro have been applied, for instance, to Fourier series. So our example suggests a wider educational approach, in order to consider a number of different experiences that give sense to mathematical language.
The main problem of the passage from finite to infinite is a cultural one, and historical issues are important in order to approach it, although, undoubtedly, the historical approach is to be considered together with other educational approaches (see: Radford, 1997): this problem requires a mediation (Gadamer, 2000, 811) and it can take into account Rortian "irony" (Rorty, 2003).

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Mario Barra (17.00, room 4)

Dynamic and innovative aspects in dynamic geometry software

An attempt will be made to demonstrate that dynamic geometry software allows us to:
- identify expository errors with limit the understanding of some complex mathematical themes;
- offer strategies for teaching so as to avoid some very common errors;
- identify and demonstrate some well-known mathematical properties in new way;
- identify and demonstrate some new mathematical properties.

Illustration of these possibilities will be sought by talking about the following topics:
- an infinitesimal method for demonstrating Pythagoras' most general theorem on the plane;
- a method for overcoming an error in understanding relating to the "law of large numbers", which has been present in 80-90% of the approximately 2,000 answers to a questionnaire which has been issued to final-year students on mathematics degree courses and to secondary school teachers since 1984;
- a method for overcoming an "error" in exposition, present in "all" the representations of the discrete distributions;

- some very simple new demonstrations of the formula of ”area of the cycloids”;
- the sums of some distributions of errors and a new way to obtain the b-splines and to demonstrate how these can generate: larger b-splines, the constants, the straight lines, the parabolas, …;
- the “pulsating tessellations" on the plane and in space;
- a new concave solid with central symmetry, which has equal and non-regular faces and show a new three-dimensional solid tessellation;
- a new non-linear transformation which is useful for determining the areas identified by “Archimede's spiral”, by the cardioid, …, and by new curves.

The conclusions will be made up of some general considerations relating to the examples presented.

M. Cerulli , J.F. Georget, M. Artigue, R.M. Bottino, Chaachoua H, M.A. Mariotti, M. Maracci, B. Pedemonte, E. Robotti, J. Trgalova (TELMA European Research Team, 17.30, room 4):
KALEIDOSCOPE: Network of Excellence

This contribution is about a research activity that is jointly carried out by six European teams belonging to the Network of Excellence Kaleidoscope (http://www.noe-kaleidoscope.org/).
Kaleidoscope is an initiative founded by the European Community (IST–507838) under the VI Framework Programme which brings together key European teams with the aim of developing new concepts and methods for exploring the future of learning with digital technologies. Within Kaleidoscope, a number of different research activities, covering a wide range of topics, have been carried out. Among these, a European Research Team (ERT) has been established to focus on the improvements and changes that technology can bring to teaching and learning activities in Mathematics. TELMA (Technology Enhanced Learning in Mathematics) ERT includes five European teams (and among these French and Italian teams) with a strong tradition in the field. Its first aim is to promote integration among such teams and to favour the construction of a shared scientific vision, the development of common projects and the building of complementariness and common priorities.
At the beginning, integration has been addressed collaboratively analysing the work of each team belonging to TELMA according to a number of different perspectives: theoretical frameworks of reference, ICT tools designed and/or employed, research methodologies, projects carried out, etc. The first phase of this collaborative work was mainly based on the descriptions provided by each team and on the analysis of some of their most representative articles. Since from this work it was clear that in order to find similarities and to clarify differences it was necessary to find some common perspectives under which to look at the different teams approaches, we decided to concentrate our analysis on three interrelated topics: the theoretical frameworks within which the different research teams face learning research in mathematics with technology, the role assigned to representations provided by technological tools and the way in which each team plan and analyse the context in which the technology is employed.
As a step toward this analysis a common methodological construct has been elaborated: the notion of “didactical functionalities” of ICT tools. This notion provides a way for comparing theoretical frames considering their manifestations in the concrete researches carried out by the different teams (e.g. the design of ICT-based learning environments, the empirical experiments in classrooms, etc.).
The notion of didactical functionalities was initially introduced in the internal report accessible from the TELMA website and then further elaborated in a contribution presented to CERME 4 (Cerulli & Al., 2005). It individuates 3 main analysis concerns: a set of features/characteristics of the tool employed, an educational goal towards which the teaching and learning activity mediated by the tool is oriented, the specific modalities of employment of the tool in the teaching and learning process carried out to reach the outlined educational goal.
Even if this notion revealed itself quite useful to improve communication among groups and led to an interesting conceptualization of the use and meaning of technological tools in educational settings, the limits of an analysis based only on published papers and written reports appeared soon evident.
For this reason, to understand, beyond a pure declarative level, the role (both implicit and explicit) played by theoretical frameworks, representations and contexts in the research work carried out, it was then decided to prepare a joint short-term project based on a cross-experimentation approach. That is, it was decided that each TELMA team would experiment, in real class settings, an ICT-based tool it had not produced but that was developed by one of the other teams. It was also agreed that the cross-experimentations would be carried

out according to jointly developed guidelines and that their results would be discussed through a collaborative activity.
In this contribution we briefly present cross-experimentations as a methodology to approach collaboration among research teams and we present some results from such experimentations focusing, in particular, on how theoretical frameworks have influenced the design and implementation of the class experiments. We focus specifically on the work and frames of the Italian and French teams involved.

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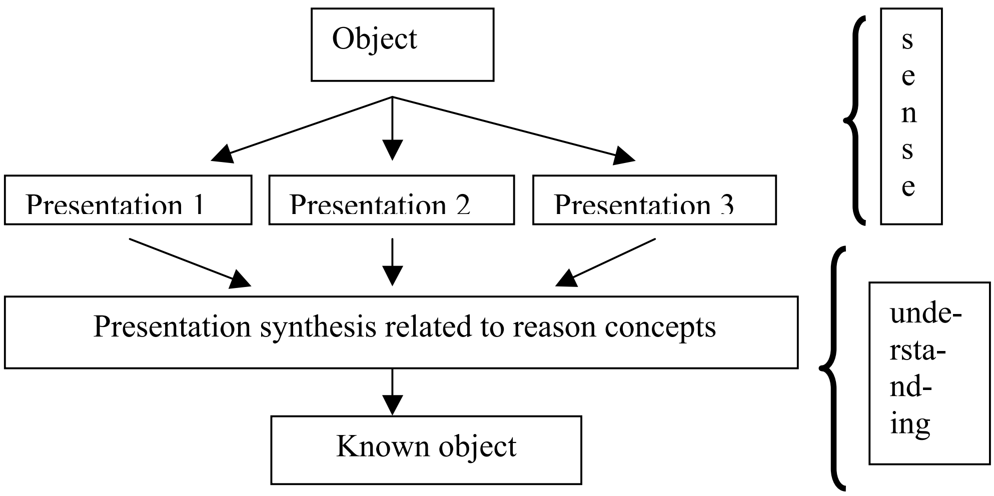
Bruno D'Amore, Martha Isabel Fandiño Pinilla (17.30, room 1)

How the sense of mathematical objects changes when their semiotic representations undergo treatment or conversion

1. Preliminary remarks. It often happens, at any school level, in mathematical situations that can also be very different between each other, that we are surprised by a statement that suddenly reveals a missed conceptual construction regarding topics that instead appeared thoroughly acquired.
We will give a roundup of examples that we found in the past years and we will try to give one of the possible explenations of this phenomenon, analysing in particular an example.
We will refer to Radford (2004) where one can find this diagram that we appreciate because of its attempt to put in the the right place the ideas of *sense* and *understanding.*



Note how the sense allows to give different "presentations" of the same object , whereas the understanding allows to say that the synthesis of theses presentations leads to the understanding of the object.

2. Mathematical object, its shared meaning, its semiotic representations: the story of an episode

2.1. The episode. We are in a primary school 5th grade class and the teacher has given a lesson, in an a–didactic situation, on the first elements of probability, allowing students to construct, at least through some examples, the idea of "event" and of "probability of a simple event". As an example, the teacher lets the students use a normal six faces dice, analysing the casual outcomes under a statistic point of view. At this point the teacher proposes the following exercise: *Calculate the probability of the following event: outcome of an even number when we toss a dice.*

The students, during a group discussion and mainly sharing practices under the direction of the teacher, decide that the answer is expressed by the fraction $\frac{3}{6}$ because the «possible outcomes of the toss are 6 (at the denominator), whereas the outcomes that make the event true are 3 (at the numerator)».

After institutionalizing the construction of this knowledege, the teacher, satisfied by this effective experience, relying on the fact that this result has been obtained rather rapidly and on the fact that students have shown a high skill in handling fractions, proposes that, being $\frac{3}{6}$ equivalent to $\frac{50}{100}$, the former probability can be exprerssed by the writing 50%, that is very expressive: it means that half of the probability to verify that event is compared with the general possible events, taken as 100. Somebody notes that «also the [fraction] $\frac{1}{2}$ works»; the proposal is validated through the statements of the proponent, it is rapidly accepted by everybody , and it is once again institutionalized by th teacher.

2.2 Semiotic analysis. If we analyse the different semiotic representations that emerge from this activity, regarding the same event: "outcome of an even number when we toss a dice", we find at least the following:

- natural language semiotic register: probability for the outcome of an even number when we toss a dice.
- fraction language semiotic register: $\frac{3}{6}$, $\frac{50}{100}$, $\frac{1}{2}$
- percentage language semiotic register: 50%

2.3 *The sense shared by different semiotic representations.* Each of the above semiotic representations is the signifier that is *down the plain* of the same signified wich is *at the top* (Duval, 2003). The shared "sense" regarding what was being constructed was always identically present and therefore the displayed mathematical practice, and thus described, has lead to semiotic transformations whose final results have been easily accepted:

- conversion: between the semiotic representation expressed in the natural language and $\frac{3}{6}$
- treatment: between $\frac{3}{6}$, $\frac{50}{100}$ and $\frac{1}{2}$
- conversion: between $\frac{50}{100}$ e 50%.

2.4 Follow- up of the episode: the loss of the shared meaning because of the semiotic transformations. At the end of the session we propose the students the fraction $\frac{4}{8}$ and we ask if also this fraction represents the same event exoplored before, since it is equivalent to $\frac{3}{6}$. *The unanimous and convinced answer is negative.* The teacher himself who before had directed with confidence this situation, declares that «$\frac{4}{8}$ *cannot represent* that event because the faces of a dice are 6 and not 8». At the researcher's insistence to know his thought relative to this concept, the teacher declares that «there exist not only 6 faces dice but also 8 faces dice; in that case, yes, the fraction $\frac{4}{8}$ represents the outcome of an even number when we toss a dice».

2.5 We examine what is going on in the cassroom from a semiotic pont of view

- The mathematical object (signified) O_1 must be represented: the probability of the outcome of an even number when we toss a dice;
- we give it a *sense* deriving from an experience that we think being shared in a social practice that is constructed because apportioned in the classroom;
- we choose a semiotic register r^m in which we represent O_1: $R^m_i(O_1)$;
- we perform a treatment: $R^m_i(O_1) \rightarrow R^m_j(O_1)$;
- we perform a conversion: $R^m_i(O_1) \rightarrow R^n_h(O_1)$;
- we interpret $R^m_j(O_1)$ recognising in it the mathamatical object (signified) O_2;
- we interpret $R^n_h(O_1)$ recognising in it the mathematical object (signified) O_3

What is the relationship between O_2, O_3 and O_1?

We can recognize an identity; therefore this means that there is at the origin a previous knowledge as a basis on which the identity can be established.

We can in no way recognize an identity, i.e. the "interpretation" is or it seems different, and therefore the *sense* of the initial object (signified) O_1 is lost.

Nadia Douek (17.00, room 1)

Language, experience of activity, and theorisation at early stages

In the expert's development of mathematical knowledge, we can recognise phases in which some knowledge, procedures and competencies are organised into a theoretical construction. We wish to describe, from a developmental point of view, elements of such an evolution at its early stages, and to put in light the role of various language and semiotic activities to favour similar movements between "activity" and "theorisation" in primary school context, in the experience fields didactical setting. The description of the dynamic is inspired by our interpretation of the Vygoskian dialectic of scientific knowledge/ everyday knowledge: we will use it both to clarify what we intend by "theorisation" and as a model to describe the dynamic.

Theory and theoretical thinking are difficult for students to access, especially in mathematics. Because they are part of mathematical activity and structure, we need to 1) characterise them in an epistemological and cognitive perspective, and 2) to describe their relation to activity in a more general sense. The aim is to discuss a way to introduce very young students to some of their relevant aspects. So we will also 3) make some hypothesis concerning favourable conditions, from a didactical point of view.

Vygotsky's elaboration of the scientific concepts/ everyday concepts dialectic is a rich and powerful tool to address the first two questions. We will present our interpretation of this dialectic, as a tension between tendencies rather than between determined static objects. Vergnaud's definition of concepts will allow us to refine our analysis when following such a dialectic.

In accordance with this theoretical framework, we will consider the transition of what could be described as common knowledge and ordinary thinking processes, to theories and theoretical thinking. This dynamical perspective will enable us to elaborate a characterisation of "theorisation", taking in consideration epistemological, semiotic and cognitive perspectives. We will also give a description of a "activity / theorisation dynamic". Vergnaud's components of concepts can allow us to makes visible activity (including semiotic), and relation-to-"reality", activity aspects of concepts.

Thus we have means to better understand the nature of this transition. These elements will be used as signs to detect theorisation movements in the experimental situation.

The experience field didactical theory, which is coherent with the vygotskian theoretical construction on various levels, will be an efficient theoretical frame to deal with the third question. We will rapidly present this theory, and we will build our didactical hypothesis upon the analysis of an example.

We will consider a didactical sequence where some early steps of transition to theorisation were accomplished. It took place in Ezio Scali and Nicoletta Sibona's class in Piossasco (near Torino, Italy), where teaching/learning situations are developed in the experience fields didactical setting.

We will put the "signs" of theorisation movements into evidence. Their linguistic and semiotic aspects will be analysed, both as signs and as means of theorisation.

A transition towards theoretical thinking implies a complex and demanding cultural and cognitive evolution. What may be considered as favourable conditions? How these movements can then become objects of intentional mediation by the teacher? In this didactical context, as in the example, the problems are frequently posed with the aim of pushing students to adopt a theoretical position. Therefor, we will enhance some important characteristics of the didactical setting, which we considered crucial for favouring the theorisation movements.

Conclusion conjectures about the means and didactical choices that favoured such movements will concern:

- Building up the situation and developing favourable activities;
- teachers' handling of context, tools and more theoretical elaboration;
- the nature of problem posing, and of questions;
- a graduation of materiality / virtuality components of the resolution procedures;
- the role of semiotic activities, and specifically language ;
- didactical contract;
- some crucial elements of the routines of the didactical setting.

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Jean-Baptiste Lagrange (16.30, room 4)

The Casyopée project: computer symbolic computation for students' better access to algebraic notation and rich mathematic

A common concern by mathematics educators is that secondary students generally get little acquaintance or proficiency in paper/pencil algebraic representation. Many students have then no means to access rich mathematics that could help them to understand the role of mathematics in today's society.

Can technology be a remedy by offering new modes of representation? In this orientation, some researchers are looking for *new* technological representations easier or more motivating in themselves. For instance, using a spreadsheet could help students to represent situations and to solve problems without a direct confrontation to the algebraic notation. It could then allow an entry into algebraic thinking .

The Casyopée project (Lagrange 2005) takes another orientation. Its ambition is to use technology to provide students for means to access *existing* algebraic representations built by mathematicians in the preceding centuries. It is based on a transpositive perspective: mathematicians did not abandon the algebraic notation, but rather developed specific computer artefacts (symbolic calculation, mathematical writing systems…) to help their everyday practice of this notation.

Symbolic calculation (generally referred to as Computer Algebra Systems, CAS) is now available for everybody's computer or calculator. The Casyopée project takes into account the potentialities that CAS offers to teaching/learning, but also the difficulties of classroom use of standards CAS established by research studies like Lumb et al. (2000).

The Casyopée project starts from potentialities of CAS outlined below:

a. Going beyond mere numerical experimentation and accessing the algebraic notation.

Dynamic geometry or spreadsheet certainly offer students means for modelling siutations and experimenting but CAS offers expressive means much closer to ordinary mathematical notation and much more powerful. Computer algebra could help students to go beyond mere numerical experimentation like it does for mathematicians.

b. Focusing on the purpose of transformations rather than on manipulation.

In paper/pencil, algebraic manipulations and transformational skills are necessary in order to get a given form, possibly hiding the interest of the outcome as compared to the initial form. Basic capabilities of CAS (expand, factor…) help students to choose a relevant transformation for a given task.

c. Connecting the algebraic activities.

Students' use of CAS in an experimental algebraic activity would also help to better articulate the varied algebraic activities. Yerushalmy (1997) studies a classroom exploration of the asymptotic behaviour of functions with help of a graphing tool. Students had to link the perceptive evidence of an asymptotic line on the graph and the partial fractional expansion of the function. But the graphing tool was of no help to get the expansion and students had to use paper/pencil polynomial division. Few students could do it alone and, as it was a long process, they lost view of the goal. If students had used CAS, they could have freely explored algebraic transformations looking for one corroborating their graphical observation.

A major choice in the Casyopée project is to develop a software environment embedding algebraic transformational knowledge by way of a 'state of the art' symbolic kernel and facilities to help build and write a proof. Casyopée has three ambitions:

a. To be an open computer environment that students could master and easily link with paper/pencil mathematics,

We aim easier instrumentation and better curricular adaptation by offering symbolic capabilities easy to connect with usual secondary mathematical practice, and to encourage methods or techniques as a way to conceptualisation. Because CAS non-specificity creates obstacles to classroom integration, we choose a domain to implement these ideas: the study of real functions, including parametrical functions -or families of functions.

b. To give a clear statute to algebraic objects.

Casyopée's organisation is designed to help students to keep clear of erratic behaviour often observed in the use of standard CAS by concentrating on relevant objects in problem solving, to make sense of experimentation and to develop methods of proof. As a difference with standard CAS, which operate mainly on symbols, each object has a clear status with regard to the curriculum: real number, function, parameter… Functions are defined on IR or on reunion of intervals. While standard CAS' window is just a memory of commands and feedback, Casyopée's interface displays the objects relevant for a problem –real numbers, functions with their definition and standard forms (factored, developed…), equations, parameters- and their properties. These objects are dynamically updated like in a spreadsheet when some change is done.

c. To help to link enactive and theoretical representations in calculus.

We want to facilitate graphic and numerical exploration like with a grapher, while encouraging transition to global/meta activities by generalisation using symbolic computation. We developed a special feature -instantiating and de- instantiating parameters. Together with the dynamic organisation of objects, it is an aid to students' exploration and problem solving involving families of functions. Dynamic instantiation helps to study numeric cases and deinstantiation corresponds to generalization.

Casyopée is an evolving project, especially now inside the STREP ReMath[2]. In its present state, the students' multi representational exploratory activity is limited: students cannot build and try models of enactive phenomenon by themselves. We plan an extension to allow students to work with enactive representations of 'real world' phenomenon and to introduce more interactivity into the work on algebraic and non-algebraic representations and in the changes of representations (modelling, interpreting, converting…). This will be achieved by complementing Casyopée with a module providing 'enactive' experience of functions. Dynamic manipulation of lengths, area and volumes will be chosen as a good start to help students to build and try functional models of relationships between these.

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Maria Polo (18.00, room 1)

Analysis of the teacher position in the coaching of classroom practises: official and real curriculum

In this paper I would like to discuss some consequences of my research concerning the observation and the analysis of classroom practices, from teacher position's view point[3]. The research which I present involves two subjects of study: formation trainings[4] focused on the integration of the software CABRI into mathematics curriculum of primary and secondary school; activities about mathematics contents, especially elementary arithmetic contents, in setting up extracurricular projects at secondary school two years period[5] level. These activities, managed according to a modality which has always maintained in my researches a dialectic between core research, both theoretical and applied (Arzarello 2000), and the usual practice carried out in the classrooms, proved to be research laboratories pertinent to emergency, to the observation and to the analysis of research problems issues: *structural and temporal economy of the didactic system* (Assude, 2001; Arzarello e al, 2002) *and the structuring of the milieux* (Margolinas & Steinbring, 1994).

Keeping up with the results of Assude 's work 2001, we maintain that problems issues concerning the *structural and temporal economy of the didactic system* arise whether when we intend studying connections between official and real curriculum or when we intend analysing the integration of knowledge not involved in the official curriculum in innovative educational practices.

It's, in fact, by acting in the class that we can detect the constraints which weigh on the teacher to sound out some specificities of the *teacher position* in the working of the structural and temporal didactic system. As underlined Assude, to find these constraints observable at the real curriculum level.

> «[...] plusieurs facteurs peuvent être considérés: la conception et la gestion d'une ingénierie didactique, le choix des situations didactiques, la gestion du contrat didactique, la structuration du milieu, le problème de la genèse instrumentale (lorsqu'on travaille avec des outils), le choix des praxéologies mathématiques et didactiques, le choix des activités. (…) La gestion du temps apparaît comme un facteur important dans le curriculum réel […] La recherche d'une stabilité pour concilier la tradition et l'innovation, ou l'ancien et le nouveau, apparaît comme un des éléments de cette économie temporelle ». (Assude, 2001, pp 84-85).

In our experience we analysed teachers' work based on the construction of a series of activities focused on the CABRI integration in the mathematics curriculum and on the observation of students who are working in some activities designed and managed by the teachers' trainer . This analysis has shown on one hand the tendency to continue an official curriculum constitutive linearity
(geometrical activities reproducing the text books' axiomatic organization), we have been able to observe the reducing or the elusion of a possible instrumental genesis of knowledge (Rabaldel, 1999) and on the other hand a transparency, in the eyes of the teachers, of the distance, observable in the act of working, between the chronogenesis and the topogenesis of the knowledge as well as its pre-built and built status (Chevellard 1995). The constraints which weigh on the *teacher's position* issues of this analysis reveal an epistemology of the teacher classically known as Brousseau underlines

> «La cronogenèse historique des mathématiques montre des réorganisation successives de leur topogenèse, propre à faciliter l'attaque de problèmes nouveaux par les mathématiciens d'aujourd'hui. *Cette réorganisation* n'est pas nécessairement mieux adaptée qu'une autre à la résolution des problèmes que les élèves doivent produire aujourd'hui, mais elle *tend à s'imposer au titre de la fonction même de l'enseignement* (qui est d'adapter les élèves à la culture actuelle) » (A.A. 2005, p.34).

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[3] According to the definition given in Polo M., 2002 «Il nostro oggetto di studio è in particolare la *pratica didattica* nella quale agiscono *Insegnante* e *Alunno* assunti come *elementi* nell'analisi sistemica della *relazione didattica* che li lega al *Sapere*. (…) Nel seguito quando ci si riferirà all'insegnante ed all'alunno in quanto *elementi* del sistema si utilizzeranno le locuzioni *posizione insegnante* e *posizione alunno*.

[4] The formation has been realised according to the three strategies: ostensive demonstration , homology and accompaniment [Assude et Grugeon, 2003].

[5] In Lai, Polo 2002, we have proposed a early problematic about the contingence analysis with the help of the theoretical concept of milieux as a researcher's working tool.

In this forming experience on CABRI as well as in the integration of activities in the extracurricular projects practices (in this second case, by the analysis of the documents of the teachers' courses projects , of their students evaluation in mathematics which we have intersected with homework exercises solved by the students during the activities built and managed by teacher' trainer in class during the teacher's absence) in terms of the *structuring of the milieux* we have found according to Margolinas, 2004. that the *project level* (level +1) is favoured in the teachers' activities, whether in the preparation for courses or in the class management . Problems of other upper levels and inferior levels aren't made up consciously, and some actions which we could expect from the teacher (for example the maintaining of the devolution's process) don't appear at inferior levels of the didactic and a-didactic situation (level 0 and 1).Particularly we have identified a certain teachers' behaviour invariance compared to the possible expectation of students' behaviour : the inclination to maintain an inelasticity of the didactic relation, in a kind of short circuit of the devolution processes and of knowledge's institutionalization which are at stake in the situation, as well as a supposed naturalization of the knowledge which maintain, for students, a pre-built regulation. To what extent is that invariance a characteristic of the teacher' position?
These results *don't assume but* working trails about constraints understanding which weigh on the teacher's position and at the same time open questions concerning the problems issues integration, in the usual class practices, of knowledge' objects *not involved* in the official curriculum as is the case, at least in Italy, of the large extracurricular projects which take place in the schools.

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George Santi and Silvia Sbaragli (18.00, room 4)

Semiotic representations and avoidable miconceptions

Radford (2005) claims that the learning is an *objectification* process, etimologically a process aimed at bringing something in front of someone's view, that transorms cultural objects into objects of our consciuosness. This process of objectification is particularly difficult in mathematics since not even the most accurate ostensive gesture would be good enough to approach its objects (D'Amore 2003; Radford 2002, 2005), but it is necessary turning to representaions that Radford (2005) calls *semiotic means of objectification* related to the social practices that originate them. Thus conceptual learning takes place at the meeting point of the students' subjectivity, socially constructed semiotic means of objectification and a

social and cultural system of meanings (Radford 2000, 2005). The semiotic means of objectification are manifold and include both intellectual and sensory activities: the sensory and kinesthetic body activities, gestures, artefacts (objects, technological instruments, etc.), and semiotic registers (algebriac, figurative, etc). Mathematical objects conceptualization does not occur with only one of these possible means, because mathematical meaning is forged out by the interplay of various semiotic systems. Teachers have the delicate task of guiding and sustaining the student while ccordinating different semiotic means, that are themselves complicated and difficult to handle, to avoid that students confuse the mathematical object with one of its possible representation. This kind of misundestanding can inevitably lead to the idea of "misconception".
Within the semantic and constructive approach, the term *misconception,* proposed by D'Amore (1999) and by D'Amore, Sbaragli (2005), we distinguish "*avoidable"* misconceptions (Martini, Sbaragli, 2005; Sbaragli, 2005) that depend *directly on didactic transposition of knowledge,* since they are a direct consequence of teachers' choices; from "unavoidable" misconceptions that depend on necessary didactic choices.
In a semiotic-cultural theoretical framework, (D'Amore, 2003; Radford 2000, 2005) "avoidable" misconceptions are therefore ascribable to the school praxis "undermined" by improper habits proposed by teachers to their students. On the other hand, Zan affirms: « We can recognize that the kind of teaching received has a considerable responsibility on the formation of beliefs» (Zan,1998)
In particular, among the different kind of "avoidable" misconceptions we focus our attention on the ones that depend on the teacher's representations that are usually univocal and inappropriate. The emblematic example we identified regards the habit of indicating an angle with a "little arc" drawn between the two half lines of which it is composed; these representation is in contrast with the mathematical property that characterizes this "object": its boundlessness.
There are many researches, that deal with the different "deceits" deriving from angle's representations, carried out especially by Fishbein and his students (Fischbein E., Tirosh D., Melamed U., 1981; Tsamir P., Tirosh D., Stavy R., 1997; Stavy R., Tirosh D., 2000), bounded to deep and general reflections in mathematics education.
In particular this treatment belongs to the theory of figural concepts proposed by Fishbein (1993) that affirms that it is inconceivable in geometric reasoning to separete concepts from images; i.e., in the domain of geometry there is a set of mental entities called by the author figural concepts that cannot be conceived neither as pure concepts , nor as pure images, but they own simultaneously conceptual and figural properties. *A figural concept is therefore a mental entity controlled by a concept, that preserves its spatiality.*
It is necessary creating interaction processes between the figural and conceptual aspcts, bearing in mind that the superiority of the conceptual aspects on the figural ones allows to better interpret under a geometrical point of view the different spatial situations, related to the use of gestures and the perceptive and kinesthetic semiotic means. Therefore one of the didactic objectives to accomplish is to allow students to "see" objects allowing concepts to prevail on images in order to transfer the conceptual aspect on each single figural proposal given to them.

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Caterina Vicentini (18.30, room)

From π to aleph through the theatre: a way to avoid didactical frauds and pseudostructured assurances

During almost five years, and precisely from December 2000 to August 2005, I taught in the Italian "Scuola Media": my students were between 11 to 14 years old.
The national program I had to follow in my teaching includes the calculus of circumferences' lengths, circles' areas and cylinders' volumes. The formulas involved give rise to the irrational number π. The didactical question is: <u>what can be π in the minds of students at this age?</u> After the definition of π as the constant ratio C/d in which C is the circumference and d is the diameter, the Italian textbooks present essentially two ways of using π in exercises. Following the old textbooks π is always 3.14. This is what I call a "didactical fraud" since π is not what it seems (in this case it is presented as rational number!). Following the new ones, π isn't approximated at all. It remains a letter in the solution: if the radius is 3 centimetres, the circumference is 6π centimetres. At first I decided for this second point of view. Then, I started to notice that π was something strange in my students' understanding. The majority of them thought at the length of the above circumference as 6 centimetres. Π remained an empty symbol in their brains. They were "pseudostructured" in the sense of Sfard, that is they tended to underrate the semantic aspect and stay at the syntactical level. I was (unintentionally) contributing to their viewing mathematics essentially as a set of more or less meaningless symbols, which they should have been able to manipulate in order to succeed at school and in their life. How could I overcome this difficulty? I decided to ask them to write the exact solution followed by an approximation. If the data were given using centimetres, the approximation had to end at the first decimal position for lengths, second decimal position for areas and third decimal position for volumes. In this way we were coherent: we had the precision of the millimetre for lengths, the square millimetre for areas, the cubic millimetre for volumes. The value of π used in calculating was that given by the scientific calculator used. <u>Understanding this approximating process forced pupils to conceive an actual infinite set of natural numbers: the decimal ciphers of π</u>. It wasn't easy at all.

Some years before (school year 1998-1999), working in the same school I teach now, the Istituto d'Arte "Max Fabiani" in Gorizia, to motivate a difficult group of students, I decided to write with them a piece of theatre called Hotel Aleph inspired by the well known Hilbert metaphor concerning an hotel with an infinite number of rooms. Unfortunately this play was not performed because of the difficulties in finding colleagues available to help us in the staging.

At the beginning of the school year 2002-03, I eventually thought that playing this drama would have been both useful and funny for my young pupils. By chance, when I proposed to the Consiglio di classe[6] of the class 3 B in the Scuola Media "Del Torre" located in Romàns d'Isonzo (Gorizia) to stage the comedy, I found five colleagues ready to work at this didactical project: the Italian teacher[7], the Art teacher[8], the Music teacher[9], the Physical Education teacher[10] and the special teacher[11] helping us with disabled pupils.
Working together was really difficult and amusing at the same time. First of all my colleagues had to understand the mathematical concept and then they had to agree on the "mise en scène". We had a lot of discussions which were often philosophical. All these meetings forced me to rewrite the text completely.

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[6] The meeting of all teachers of a certain class.

[7] prof. Laura Delpin

[8] prof. Wilma Canton

[9] prof. Laura De Simone

[10] prof. Laura Valli

[11] prof. Bruno Raicovi

Communicating this work I would like to show the DVD of my pupils' performance, with English subtitles and to talk about the "mathematical backstage". During the preparation we read a short dialogue from "Discorsi e dimostrazioni matematiche sopra due nuove scienze" written by Galileo Galilei in 1638. More exactly, we studied the part concerning the bijection between natural numbers and their squares, included in the "Giornata Prima"[12]. Eventually we came to the translation of both the situations using the functional representations. "Hotel Aleph" shows the bijection between natural and odd numbers: $h(n) = 2n$, this can be seen as a line[13] drawing a graph in the Cartesian plane; Galileo example about natural numbers and their squares in the modern mathematical form gives rise to: $g(n) = n^2$, that is half of a parabola[8].
Since I was teaching to young pupils I didn't go over. I didn't give them the definition of an infinite set, as I had done with the students at the Istituto d'arte. We concluded with the same kind of perplexity Galilei had had. At the end of the theatrical text I let Marisa say: "It seems a miracle, but it is also perfectly logical!". I think that, considering the age of my students, what we did was enough. As a matter of fact, I affirm that <u>a "good perplexity" is a better source of learning than a "pseudostructured assurance"</u>.

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[12] the "First Day". "Discorsi e dimostrazioni matematiche sopra due nuove scienze" is written in the form of a dialogue between three characters: Simplicio, Sagredo e Salviati taking place on different days.

[13] even if full of holes!